OUTSTANDING ERRORS

OF

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P. BLACKSTONE, A. M., C. E.

BERLIN, WIS.
GEO. C. HICKS, PUBLISHER
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A correct nautical almanac is of unspeakable value to the astronomer, while an incorrect one makes it perilous for ships at sea. For a definite understanding of these outstanding errors the following quotations are made.

The first quotation is from the pen of Simon Newcomb, as published in Harper's Magazine July, 1902; the second was written by C. A. Young and published in The Cosmopolitan June, 1894; and the third is a production by John Elfreth Watkins, Jr., and published in the Ladies Home Journal July, 1903.

First:—

"Our own moon is one of the enigmas of the mathematical astronomer. Observations show that she is deviating from her predicted place, and that this deviation continues to increase. True, it is not very great when measured by an ordinary standard. The time at which the moon's shadow passed a given point near Norfolk during the total eclipse of May 29, 1900, was only about seven seconds different from the time given in the Astronomical Ephemeris. The path of the shadow along the earth was not out of place by more than one or two miles. But, small though these deviations are, they show that something is wrong, and no one has as yet found out what it is. Worse yet, the deviation will in all likelihood go on increasing rapidly. The mathematical problems involved are of such complexity that it is only now and then that a mathematician turns up anywhere in the world who is both able and bold enough to attack them.

"Twelve years ago a suspicion which had long been entertained that the earth's axis of rotation varied a little from time to time was verified by Chandler. The result of this is a slight change in the latitude of all places on the earth's surface, which admits of being determined by precise observations. The National Geodetic Association has established four observatories on the same parallel of latitude—one at Gaithersburg, Maryland, another on the Pacific coast, a third in Japan, and a fourth in Italy —to study these variations by continuous observations from night to night."

Second:—

"THE WABBLE OF THE EARTH'S AXIS.

"We do not refer to the long-known shifting of the direction of the axis of the earth, which produces the so-called 'Precession of the Equinoxes.' This does not in the least affect the position of the pole upon the surface of the earth, while that which we have in mind is an actual travelling of the pole over the ground, and is due to a slight change of the position of the axis within the globe itself. This, of course, manifests itself by a minute change both in the latitudes of observatories, and in the direction of meridian lines. If the pole of the earth approaches Berlin, the latitude of Berlin is necessarily increased, and at the same time the latitude of Honolulu, on the other side of the earth, is correspondingly diminished.

"The fact that such a thing is really happening was first clearly brought out in 1889 in Germany, and ever since the subject has greatly interested the astronomical world. All recent latitude observations made by methods of precision confirm the fact, and within a few months Socoloff has shown that the azimuthal observations upon the Pulkowa meridian-marks between 1880 and 1887 tell the same story. The latest results of Chandler, based upon a very thorough discussion of several thousand observations made at seventeen different observatories, between 1840 and 1893, and combined with earlier series at Greenwich (by Pond, between 1820 and 1830, and by Bradley about the middle of the last century), show that this motion is unexpectedly regular.

"It seems to be made up of two superposed revolutions of the pole from west to east; one with a period of just a year, in a circle of about thirty feet in diameter, and the other in a circle of similar size, but with a period of 428 days. As a consequence of this combination of motions, the actual annual displacement varies greatly. Once in about seven years the two practically destroy each other, and the pole remains for a time nearly stationary (as in 1885), while at intermediate epochs (as in 1890), it describes a sort of circle fully sixty feet in diameter.

"It hardly need be said that a motion so slight becomes sensible only in observations of the last degree of precision, but its discovery has already explained certain important anomalies and apparent errors in work of that class.

"As regards the cause of this peculiar motion, the theory is still more or less obscure. There can be little doubt, however, that the annual component is due, in part at least, as Lord Kelvin long ago suggested, to the course of the seasons—to the winter's deposit of snow and ice upon the northern continents, and its later return to the ocean.

As to the 428-day revolution, this seems to be a veritable 'wabble,' such as is produced by striking a spinning top. The 'blow' may perhaps consist in the annual disturbance just referred to; but the matter is not yet wholly clear."

"C. A. Young."

Third:—

"THE NORTH POLE IS CONTINUALLY MOVING.

"It is perpetually roving within the limits of a circle sixty feet in diameter. What is the North Pole to-day is not the North Pole to-morrow. The true North Pole has been known to travel more than four feet in a week, while sometimes it has required more than a month to cover a yard. Suppose that you and I were to sail from opposite points to discover this turning point. We will say that you, with your astronomic instruments, planted your flag upon the exact North Pole six months ago, and then went away. I, arriving to-day, make equally accurate calculations and plant my flag also upon the true North Pole. My flag is probably forty feet from yours, yet neither of us is in error. To-morrow the elusive little tip-top of the earth will have slipped away from both of us. And if I were to claim a building lot, the cornerstone of which was marked by this North Pole, a strange predicament would follow. I should have to place my fences upon casters and keep them continually moving in order to mark strictly my own reservation. Our Naval Observatory now employs an astronomer whose sole duty it is to keep track of the migratory and nomadic poles. And this he can do, by aid of his instruments, without leaving Washington."

These nomadic wanderings of the earth and the moon are all due to the same cause, and that cause stated verbally is very simple to comprehend. Under the condition of Kepler's first law that the radius vector of a planet revolving around the sun describes equal areas in equal times, the mean of the inverse squares or cubes of all the radius vectors of a planet is a very trifle greater than the semi-major axis of the elliptical orbit likewise taken.

This very small error not taken into consideration by the nautical almanac makers leads to outstanding errors between the computed tables of the nautical almanac for the moon and the results of observation. As determined by due mathematical demonstration and computation for the cause under consideration, the moon in orbit is in advance of its otherwise computed position, or right ascension, by angular gain of 1.06″ per year. This angular gain is due to the eccentricity of the moon's orbit.

Ptolemy, the celebrated astronomer, made astronomical observations at Alexandria, Egypt, about the year A. D. 140. His record of observations is known by name *Almagest*. In this book there is recorded

the time of occurrences of eclipses observed by himself and Hipparchus and other ancient astronomers.

From the data of the *Almagest* for the ancient eclipses our modern mathematical astronomers by computation have determined that the moon has gained in orbital distance from time of Ptolemy to the present date about one degree.

As determined by the lately deceased generation of mathematical astronomers, the moon has gained in orbital distance about one-half of a degree, or 1836″ since the time of Ptolemy to the present date, due to the cause that the earth's eccentricity of orbit is very slowly diminishing. This gain is accelerating, commencing at the time of Ptolemy, 6″ for the first century, 24″ for the second century, and so on for 1750 years to present date. The present generation of astronomers accept this result as very nearly true. Thus accepted, there yet remains 1836″ to be accounted for. 1.06″ multiplied by 1750 equals 1855″, making a total gain of a trifle over one degree since the time of Ptolemy.

Due to the cause under consideration, the perihelion of the planet Mercury is ahead of its otherwise computed position by angular distance 37.67″ for a period of one century. The distinguished French astronomer Leverrier determined by observation that the perihelion of this planet Mercury is in advance of its computed place by about 40″ per century, and he concluded that this discrepancy was due to a group of small planets revolving nearer the sun. Persevering search has been made for these little planets, with the result that not one has been discovered.

The Newtonian law of attraction is a theory derived from observation and has been well tested. Thus taken, all new mathematical deductions from that theory must reasonably well meet the results of observation.

Again, due to the cause under consideration, the perihelion of the planet Venus is ahead of its otherwise computed position by angular distance .4908″ for a century. Likewise the Earth is ahead .740″, planet Mars 2.19″ ,Jupiter .1790″, Saturn .0839″, Uranus .0235″

and Neptune .0024″, each per century. Excepting in the cases of the moon and planet Mercury the very small orbital gains of the other planets have not yet become apparent to observation.

From the conclusions reached by our mathematical investigation it is evident that for orbits of ellipses, the periodic times of the planets of our solar system are very slowly and unceasingly becoming shorter. Stated in other words, the solar and the stellar systems are unstable. Due to this instability a collision between two stars is remotely liable to occur. In fact, from observation, about once in twenty years a burst-out of light in the belt of the Milky Way of the heavens, like a new star, has been seen. After a few days or weeks this seemingly new star scattered into nebula from heat generated by collision. Thus it is from living star to dead star, and from dead star to nebula, and from nebula to new star solar system under new conditions. Otherwise stated it is from death to life again, or from sleep to wake again into new environments.

During a single revolution of a planet around the sun, the planet's motion for its mean gain is irregular. For the perihelion radius vectors the gain is greater than the mean and less for the aphelion. The mean density of Mercury, for instance, has been computed from a certain irregularity of motion, and it has been inferred that this planet's mean density is 6.85 specific gravity as due to the attraction of Venus and other planets. A large part of this irregularity is produced by the cause under consideration. And when truly determined it doubtless will be found from correct computation that the density of this planet is less than that for Mars.

Measured by the diameter of a circle, the north as well as the south pole of the earth each revolves about in a so-called circle 28 feet in diameter during an anomalistic revolution of the earth, commencing at the meridian of perihelion and returning to the point of beginning at the end of the year. This is caused by the attraction of the sun on the earth, and due to varying distance and the spherodial figure of the earth.

For the moon's perigee taken at longitude of winter or summer solstice for one anomalistic revolution of the moon, either pole of the

earth revolves in a so-called circle 15 feet in diameter. And for perigee taken at longitude of vernal or autumnal equinox the diameter of circle becomes zero. These changes for diameter of so-called circle varying from fifteen feet to zero are for the moon's orbit taken continuously in the plane of the ecliptic. The perigee makes a complete revolution from west to east in 8.8 years, and the time of change from fifteen feet maximum to zero maximum is 2 $\frac{1}{5}$ years.

Due to the inclination of the moon's orbit to the plane of the ecliptic, for longitude of perigee the same as already taken for the moon, during a period of 426 days (a period in our demonstration fully explained), for perigee near the longitude of winter or summer solstice, either pole of the earth revolves in a sort of a circle 27 feet in diameter. And for perigee near longitude of vernal or autumnal equinox the diameter of circle becomes 20 feet.

For the three superposed revolutions of either pole of the earth, the due result for the outstanding correction to be made to the nautical almanac, as otherwise computed for a certain day, can be determined only by due computation. For some certain dates, however, an approximation can be made. For periods 365 $\frac{1}{4}$ days and 426 days there is a seven years period for a so-called diameter of circle, 55 feet for the middle and zero for the beginning or the end of the period. It is evident that the requisite diameters of circles for the anomalistic monthly period would somewhat modify results 55 feet and zero. After making the due correction 55 becomes 59 feet. This monthly period explains the "veritable 'wabble,' such as is produced by striking a spinning top," as C. A. Young states it.

The expression *so-called circle* is explained in our mathematical demonstration. To determine the changes in the direction of meridian lines the dimensions of this so-called circle must be known.

These three results, taken independently, for outstanding corrections to latitude for all places on the earth's surface or declination for the moon or other celestial body, are due to one of the two component forces causing precession. As the resultant of the two forces producing precession is called precession, which is nearly four, or exactly 90° divided by the angle of obliquity of the ecliptic, times

greater than the component already considered for latitude or declination, the outstanding corrections for longitude or right ascension are about four times greater than for latitude or declination.

As the astronomical clock is daily adjusted so as to keep time made by daily observation on the sun, this line of corrections for longitude or right ascension does not become materially apparent to observation except in the case of the moon.

There can be no change of polar axis of our globe within the globe itself unless a mountain range or a continent rises or sinks.

It is evident from this discussion for right ascension or longitude that the nautical almanac should be corrected for the moon for gain of 1.06″ per year, and vibrations of meridian varying from zero to nearly four times 60 feet, or 240 feet as measured on the surface of the earth.

For the determination of stellar parallax or distance, a correct nautical almanac or ephemeris is very essential. In many cases the stellar parallax is less than the outstanding errors of the ephemeris. As a sample of incongruity in results obtained by different astronomers, take the polar star. The table following is taken from a table for many stars as published by Simon Newcomb in his Popular Astronomy:

STAR'S NAME	ASTRONOMER AND DATE	PARALLAX	PROBABLE ERROR
Polar Star	Lindehau, from R. A.'s, 1750-1816	0″.144	
	W. Struve, Dorpat, 1818-'21	0 .075	
	Struve and Preuss, from R. A.'s, 1822-'38	0 .172	
	Lundahl, from Dorpat declinations	0 .147	±.030″
	Peters, from declinations, 1842-'44	0 .067	
	Lindhagen	0 .025	

It is evident from the results of computation and observation that the earth is a *wiggly* foundation for an observatory; but when the astronomer knows all about the wiggles he can correct his observations accordingly.

Due to the cause under consideration, the angle of obliquity of the ecliptic is gradually decreasing at the rate of 1″ in 59 years. In the

past this rate of decrease was greater. For instance, when this angle of obliquity was 45 degrees only 43 years was required for 1″ decrease. For the future a greater number of years than 59 will be required for 1″ decrease.

Other inferential conclusions might be drawn here, but the reader can readily make such deductions for himself while reading our mathematical demonstration.

Usually, new conclusions are not reached by using old methods. Our method is by analysis. Thus each factor of the chain of demonstration leading to the final product can better be examined and corrected by the author and the full purport of the reasoning more readily grasped by the reader.

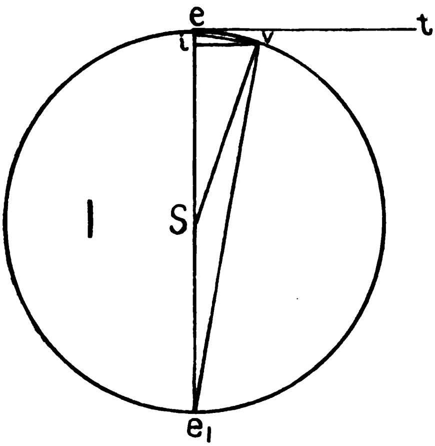
The outstanding errors, which we determine to the correctness of one ten-thousandth of a second of angle, are dropped by the ordinary methods as of value immaterial. How then can one obtain these outstanding corrections by the modern and standard rules?

Precession and nutation are so closely related to these outstanding corrections that we give a new demonstration for precession which contains some new factors from which to make deductions. Besides this we desire to familiarize the reader with our system of work before he enters into the new field of investigation.

THE MOMENT OF INERTIA.

I. Let S be the mean distance from center of sun to that of earth, and M the mass of the sun and m that for the earth. Also let g equal the attraction of the earth on a particle at the earth's surface. Then the attraction of the sun on particle at the center of the earth, the earth's radius and mass each taken unity, becomes

$$\frac{M}{S^2} g.$$



In Fig. 1, S is for location of center of sun, and e that for the earth's center. Line e v or v represents the velocity of revolution of particle at e around center S measured in tangent e t, chord or arc e v. The moment of force of particle e of mass dm for dt time in direction of tangent or arc of orbit is v dm. Take line i v perpendicular to radius S e (S) and represent distance i e by g, or differential dS. Then dS divides radius S into an infinite number of equal

parts, and the moment of inertia for particle e for distance g in direction e S becomes

$$g\ dm \text{ or } dS\ dm.$$

For radius S, circumference of revolution is $2 \pi S$. Take velocity of revolution $2 \pi dS$, then radius S and circumference of revolution each are divided into the same number of parts. For combined action of these two forces on particle e, the ratio of moment of force to moment of inertia is $2 \pi\ d\ S\ dm$ to $g\ dm$ or $d\ S\ dm$, or 2π to 1.

For constant g take point i instead of S for center of revolution, then the circumference of revolution for radius g becomes $2 \pi g$, equal to velocity v. Let t be the time of revolution about center S, then dt measures the time for velocity v or the time for a complete revolution of particle e for radius g. Thus taken particle e is at point e at the beginning and at the end of dt time instead of at point v at the end of this dt time. During this dt time the moment of force for a revolution is v dm and the moment of inertia is g dm. The moment of force v dm measuring the force required to move particle e from point e to point v is counterbalanced by the moment of inertia required to move particle e from point e to point i. As this counterbalance of these two forces signifies their equality we must put

$$g\ dm = v\ dm = 2 \pi\ dS\ dm.$$

And for time t, or a complete revolution

$$\text{(a)} \qquad g\ dm = 2 \pi\ S\ dm.$$

2. In Fig. 1, right-angled triangle e_1 v e is similar to right-angled triangle v i e.

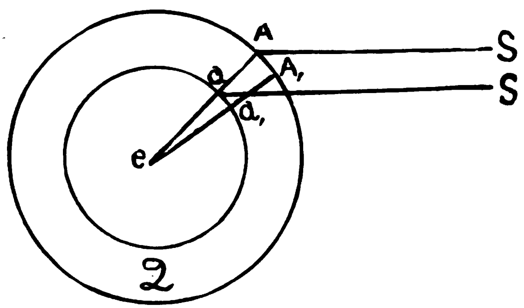
Thus taken for dt time

$$\text{(b)} \qquad v^2 = 2\ S\ g.$$

In this equation for constant g, v varies inversely as $\sqrt{S}$.

Expressions (a) and (b) are each true when duly interpreted.

3. In Fig. 2, let e be the center of the earth and particle A at the surface having radius A. Also let a represent an interior particle with radius a. Lines A S and a S extending in direction to the sun are materially parallel. These points and lines are taken in the same plane and the center of the earth taken relatively as a fixed center of rotation. The attraction of the sun on particle A rotates it around center e in direction A A_1 and likewise particle a in direction a a_1. For differential angle A e A_1 it is required that particle A shall rotate



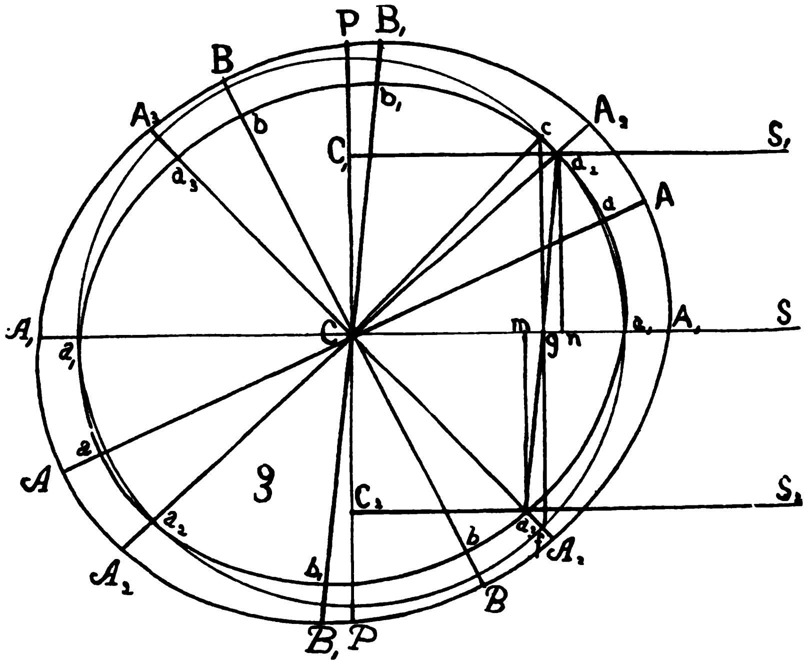
over arc A A_1, and a over arc a a_1 in time dt. Under these conditions the moments of force for the rotation of these two particles vary as arcs or radii.

(c) Under the condition of expression (b) the moments of inertia for the two particles vary as arcs or radii squared.

Expression (a) holds true for constant distance, and (b) and (c) for variations from constant or standard distance.

THE PRECESSION OF THE EQUINOXES.

4. In Fig. 3, the outer ellipse having center C and major axis A $\mathcal{A}$ and minor or polar axis B $\mathcal{B}$ represents a section of the earth situated in the plane of the earth's polar axis and the sun's center with diameter A_1 $\mathcal{A}_1$ in the plane of the ecliptic having direction to the sun's center, and B_1 $\mathcal{B}_1$ is conjugate to A_1 $\mathcal{A}_1$. The inner ellipse is con-



structed similar and lettered in small type and semi diameter C a (a) varies from zero to semi axis A. Diameter P $\mathcal{P}$ is perpendicular to A_1 $\mathcal{A}_1$. This (Fig. 3) has a circle of radius a_1 C. Line c g f parallel to diameter P $\mathcal{P}$ is a double ordinate to the circle, and line a_2 g a_3

taken parallel to diameter $B_1\ \mathcal{B}_1$ is a double ordinate to the inner ellipse. Lines a_8 n and a_8 m are to be taken perpendicular to diameter $A_1\ \mathcal{A}_1$. For the second and third quadrants conceive like construction. Lines $C_1\ S_1$, C S and $C_2\ S_2$ extend in direction to the sun's center, and thus taken due to sun's small parallax these lines are materially parallel. By construction $C_1\ a_2$ equals C n and $C_2\ a_8$ equals C m.

5. Let sun's distance measured from the center of the earth be S. Then distance from a_2 in line $a_2\ S_1$ to sun's center becomes S—C n, and likewise for a_8, S—C m.

As already taken

M = mass of sun.

m = mass of earth.

$\frac{M\ g\ dm}{S^2}$ = sun's attraction on particle C.

The attraction of the sun then on particle a_2 of first quadrant is

(d) $$\frac{M\ g\ dm}{S^2\left(1-\frac{Cn}{S}\right)^2} = \frac{M\ g\ dm}{S^2}\left(1+\frac{2\ \overline{Cn}}{S} + \text{immaterial}\right).$$

Likewise for particle a_8 of fourth quadrant

(e) $$\frac{M\ g\ dm}{S^2}\left(1+\frac{2\ \overline{C\ m}}{S}\right).$$

Results (d) and (e) are for moments of inertia.

6. Let v = angle $c\,C\,g$,

v_1 = angle $a_2\,C\,n$,

and

v_2 = angle $a_8\,C\,m$.

For fixed center C, force (d) acting on particle a_2 in direction $a_2\,S_1$ has a component force (d) sin v_1 acting at n in direction $a_2\,n$ tending to rotate the earth. Likewise force (e) acting on particle a_8 has component acting at m in opposite direction a_8 m.

Due to the opposite directions of action of these two components the resulting force acting in direction a_2 n becomes

$$(d)\sin v_1 - (e)\sin v_2.$$

This result is for two particles, one each from first and fourth quadrants. It is likewise for third and second quadrants acting in same direction of rotation.

For a single particle for the four quadrants the expression becomes

$$\frac{(d)\sin v_1 - (e)\sin v_2}{2}.$$

By due substitution for (d) and (e) the expression becomes

$$\frac{M\ g\ dm}{S^3}\left(\overline{C\,n}\sin v_1 - \overline{C\,m}\sin v_1.\right).$$

7. As already demonstrated for statement (c) to change from moment of inertia to moment of force radius squared must be taken for radius. Thus taken the expression becomes

$$\frac{M\ g\ dm}{S^3}\left(\overline{C\,n}^2\sin v_1 - \overline{C\,m}^2\sin v_2\right).$$

8. For rotation around center C, C n and C m represent radii of circles. The force of the above expression acts at n in direction perpendicular to C n or in circumference of circle. $\overline{Cn}^2$ and $\overline{Cm}^2$ expressed in units of circumference become $\frac{\overline{Cn}^2}{4\pi^2}$ and $\frac{\overline{Cm}^2}{4\pi^2}$. Thus taken the expression becomes

$$\frac{M\ g\ dm}{4\ \pi^2\ S^3}\left(\overline{Cn}^2 \sin v_1 - \overline{Cm}^2 \sin v_2\right).$$

9. For period of time one year the length of the earth's orbit is $2\ \pi$ S, and per condition of result (a) already demonstrated the expression becomes

$$\frac{M\ g\ dm}{2\ \pi\ S^2}\left(\overline{Cn}^2 \sin v_1 - \overline{Cm}^2 \sin v_2\right).$$

10. For particles a_2 and a_3 of first and fourth quadrants attraction (g) varies inversely as $\left(1 + \frac{2\ C\ n}{S}\right)^2$ to $\left(1 + \frac{2\ C\ m}{S}\right)^2$ or as inverse square of distance, or $\sqrt{g}$ varies inversely as distance.

Velocity in orbit varies as inverse square root of distance.

In terms then of gravity

Velocity varies inversely as $\left(1 + \frac{2\ Cn}{S}\right)^{1/4}$ to $\left(1 + \frac{2\ Cm}{S}\right)^{1/4}$ or inversely as fourth root of distance.

Due to the smallness of terms $\frac{2\ Cn}{S}$ and $\frac{2\ Cm}{S}$ materially true

$$\left(1 + \frac{2\ Cn}{S}\right)^{1/4} = 1 + \frac{Cn}{2\ S}.$$

$$\left(1+\frac{2\ Cm}{S}\right)^{\frac{1}{4}}=1+\frac{Cm}{2\ S}.$$

Terms 1 of these two results have already been eliminated by subtraction from expressions (d) and (e), and as $\frac{2\ Cn}{S}$ is 4 times greater than $\frac{\overline{Cn}}{2\ S}$ our general expression becomes

$$\frac{M\ g\ dm}{8\ \pi\ S^2}\left(\overline{Cn}^2 \sin v_1 - \overline{Cm}^2 \sin v_2\right).$$

Factor $\frac{M\ g\ dm}{S^2}$ is for further investigation a useless incumbrance. For this reason the expression may be taken

(f) $$\frac{\overline{Cn}^2 \sin v_1 - \overline{Cm}^2 \sin v_2}{8\pi}.$$

11. Radius Cc of the circle equals Ca_1 or (a_1) of the ellipse. For the circle, ordinate

$$c\,g = f\,g = a_1 \sin v.$$

For the ellipse, ordinate

$$a_2\,g = a_3\,g = b_1 \sin v.$$

Let $90^\circ \pm \eta$ be the angle of diameter $A_1\ A_1$ to its conjugate $B_1\ B_1$, then angle $g\ a_2\ n$ or $g\ a_3\ m$ equals angle η.

Eccentricity $e^2 = \frac{a^2 - b^2}{a^2}$ or otherwise stated for order e^2

$$b = a\left(1 - \tfrac{1}{2} e^2\right).$$

In case of the earth sin. tan. and arc of angle η are materially equal and likewise for cos. and radius.

Likewise let

$$E^2 = \frac{a_1^2 - b_1^2}{a_1^2}.$$

$$b_1 = a_1 (1 - \tfrac{1}{2} E).$$

Let acute angle A_1 C A be represented by θ varying from zero at points of equinox to angle of obliquity of the ecliptic for points of solstice.

True for the second order of eccentricity

$$\sin \eta = e^2 \sin \theta \cos \theta.$$

$$\cos \eta = 1.$$

If greater accuracy is required it can be obtained by the usual system for correction of outstanding error.

$$a_2 \, n = a_3 \, m = b_1 \sin v.$$

$$C \, g = a_1 \cos v.$$

$$g \, n = g \, m = b_1 \sin v \, e^2 \sin \theta \cos \theta.$$

$$\overline{C \, n}^2 = a^2_1 (\cos^2 v + 2 \sin v \cos v \, e^2 \sin \theta \cos \theta).$$

$$\overline{C \, m}^2 = a^2_1 (\cos^2 v - 2 \sin v \cos v \, e^2 \sin \theta \cos \theta).$$

$$\sin v_1 = \sin v \, (1 - \sin v \cos v \, e^2 \sin \theta \cos \theta).$$

$$\sin v_2 = \sin v \, (1 + \sin v \cos v \, e^2 \sin \theta \cos \theta).$$

By due substitution our general expression (f) becomes

$$\text{(g)} \qquad \frac{a_1^2 \, e^2 \sin \theta \cos \theta \, d \, m}{4 \pi} (2 \sin^2 v \cos v - \sin^2 z \cos^3 z).$$

The difference between $a^2 e^2$ and $a_1^2 e^2$ is of order e^4, and expression (g) may be taken

(h) $$\frac{a^2 e^2 \sin\theta \cos\theta \, d m}{4\pi} (2 \cos v - 3 \cos^3 v + \cos^5 v).$$

12. Angle v varies from zero to 90 degrees. For radius unity angle v varies from arc length zero to $\frac{\pi}{2}$. Let arc $\frac{\pi}{2}$ be divided into $\frac{v}{d v}$ number of equal parts. The length of each part then becomes dv. Under such conditions to find the mean value for cos v expression cos v dv can be used for cos v. Then expression

$$2 \cos v - 3 \cos^3 v + \cos^5 v \text{ becomes}$$

$$(2 \cos v - 3 \cos^3 v + \cos^5 v) \, d v.$$

Let the mean value of cos v be represented by $\cos \overline{v}$ then $\cos \overline{v}$ can be taken as a constant.

$$\cos v = 1 - \frac{v^2}{2} + \frac{v^4}{2.3.4} -, \text{ etc.}$$

$$\int \cos v \, dv = \int \left(1 - \frac{v^2}{2} + \frac{v^4}{2.3.4} -, \text{ etc.}\right) dv.$$

$$v \cos \overline{v} = v - \frac{v^3}{2.3} + \frac{v^5}{2.3.4.5} -, \text{ etc.}$$

$$\sin v = v - \frac{v^3}{2.3} + \frac{v^5}{2.3.4.5} -, \text{ etc.}$$

As expressions for sin v and for $v \cos \overline{v}$ are identical

$$v \cos \overline{v} = \sin v.$$

$$\cos \overline{v} = \frac{\sin v}{v}.$$

For arc of a quadrant

$$\cos \overline{\mathrm{v}} = \frac{2}{\pi}.$$

$$\cos^3 \overline{\mathrm{v}} = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \cos^3 \mathrm{v}\, d\mathrm{v} = \frac{4}{3\pi}.$$

$$\cos^5 \overline{\mathrm{v}} = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \cos^5 \mathrm{v}\, d\mathrm{v} = \frac{16}{15\pi}.$$

By due substitution general expression (h) becomes

$$\text{(i)} \qquad \frac{4\, a^2\, e^2 \sin\theta \cos\theta\, dm.}{15\, \pi^2}$$

13. This expression represents the force caused by the attraction of the sun on a particle of the earth in plane of Fig. 3, or any other plane cutting the earth thereto parallel. This expression is developed with reference to two co-ordinates situated in the same plane, and this force tends to lessen the angle of obliquity of the ecliptic and would if not counteracted by the force of the earth's daily rotation.

14. For the third co-ordinate the earth may be taken as a sphere of radius r and taken as already taken of uniform density.

The plane of Fig. 4 is perpendicular to the plane of Fig. 3, each Figure having a common center C. Thus constructed diameter B D and chord b d are each perpendicular to the plane of Fig. 3 and therefore parallel to each other and thus making every point in chord b d equi-distant from diameter B D. To obtain then differential expression for a single particle of the earth taken of uniform density and this particle taken under the conditions of expression (i) it remains to find differential expression for any particle in chord b d for revolution about axis B D.

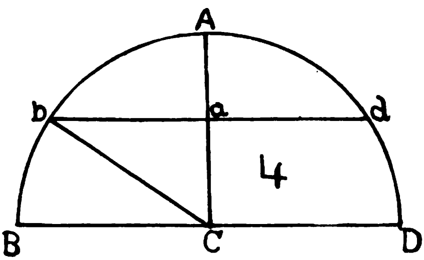
Let b A d be a segment of revolution around diameter B D, and let semi-chord a b be represented by radius r into y (r y), y being the cosine of angle C b a.

Chord b d $= 2$ r y.

C a $=$ r $(1 - y^2)^{1/2}$.

$\overline{\text{C a}}^2$ or a^2 of expression (i) $= r^2 (1 - y^2)$.

The volume generated by revolution of segment b A d is



$\frac{1}{6} \pi \overline{\text{b d}}^3 = \frac{4}{3} \pi r^3 y^3$.

The differential of this expression is

$\frac{4}{3} \pi r^3 3 y^2 dy$.

$\frac{4}{3} \pi r^3$ is the volume of the earth and may represent its mass (m) for uniform density.

$dm = 3 m y^2 dy$.

To make the differential expression comply with requirements of expression (i) for all particles located in chord b d, $r^2 (1 - y^2)$ must be made a factor. Thus the expression becomes

$$3\ m\ r^2\ y^2\ (1 - y^2)\ d\ y.$$

$$\int_0^1 3\ m\ r^2\ y^2\ (1 - y^2)\ dy = \frac{2}{5}\ r^2\ m.$$

Expression (i) thus becomes

$$\text{(j)} \qquad \frac{8\ r^2\ e^2 \sin\theta \cos\theta\ m}{75\pi^2}.$$

This expression represents the force of rotation of the earth taken of uniform density for any value for angle θ, as caused by the attraction of the sun, acting on the component particles of such earth for time one year.

16. Let Φ equal angle of obliquity of the ecliptic. Angle θ of the expression varies from zero at points of equinox to angle Φ at points of solstice. Let z be any arc, for radius unity, in the plane of the ecliptic measured from vernal equinox to summer solstice, and likewise stated in words equivalent for a revolution of the earth in complete orbit. Arc z under the assumed condition of constant distance to the sun varies uniformly from day to day. Take arc z the hypothenuse of a right-angled spherical triangle, having angle Φ at point of vernal equinox and side θ opposite angle Φ. Let b, a corresponding arc of the equator, represent the base of the triangle and angle B opposite. The distance from any point in arc b, measured in perpendicular to the intersecting line of the planes of the equator and ecliptic is sin b. And as any force applied at any point in arc b to rotate the earth on axis of this intersecting line varies in result of rotation as length of line sin b, the factor $\sin\theta \cos\theta$ for any point of arc b from vernal equinox to summer solstice differentially expressed becomes

$$\sin\theta \cos\theta \sin b\ db.$$

Solved by spherical trigonometry in terms of arc z and angle Φ this expression becomes

$$\sin\theta \cos\theta \sin b\ db = \sin\Phi \cos\Phi \sin^2 z\ dz.$$

For a quadrant

$$\int_0^1 \sin \Phi \cos \Phi \sin^2 z \, dz = \tfrac{1}{2} \sin \Phi \cos \Phi.$$

Expression (j) thus becomes

$$\text{(k)} \qquad \frac{4 r^2 e^2 \sin \Phi \cos \Phi m}{75 \pi^2}.$$

17. For the earth varying in density from "crust" to center or made up of components having radii r, r_1, r_2, etc., varying from r to zero for the radiuses of the components, and densities δ, δ_1, δ_2, etc. or masses m_1, m_2, m_3, etc. thus making a total mass m. Thus taken

$$m = m_1 + m_2 + m_3 + \text{etc.}$$

It is evident that some terms m sub can be taken zero or with minus sign and this equation remain true.

Let e^2, e_1^2, e_2^2, etc., be computed eccentricities of the components for the earth as one solid body, taken in due order of the radii.

Under such conditions $r^2 m e^2$ of our general expression becomes

$$r^2 m_1 e^2 + r_1^2 m_2 e_1^2 + \text{etc.}$$

Divide this series of terms by $r^2 m e^2$ and let the quotient be Q.

Thus our general expression (k) becomes

$$\text{(l)} \qquad \frac{4 r^2 m e^2 \sin \Phi \cos \Phi Q}{75 \pi^2}.$$

18. The force represented by this expression tends to rotate the earth on line of intersection of planes of equator and ecliptic. But due to the force of the earth's daily rotation on polar axis these two forces combine and produce as a resultant, precession. As caused by the sun and the moon the precession period is about 25,800 years. As precession is the resultant of two forces it is evident that the con-

stant component of daily rotation need not be considered. And the earth can thus be taken as rotating on axis of intersection line of planes of equator and ecliptic as permanent axis for the component under consideration.

19. For different central bodies the force of precession varies directly as mass and inversely as cube of distance.

The sun's mass is M for the earth's mass taken unity. Let the moon's mass be $\frac{1}{n}$, then the moon's mass is $\frac{1}{M\,n}$ of the sun's.

Let lunar mean distance be $\frac{1}{L}$ of sun's, then for directly as mass and inversely as cube of distance the result is $\frac{L^3}{M\,n}$.

Providing the mean inclination of the moon's orbit be taken Φ then expression (1) for both sun and moon becomes

$$(m) \qquad \frac{4\,r^2\,m\,e^2\,(M\,n + L^3)\sin\Phi\cos\Phi\,Q}{75\,\pi^2\,M\,n}.$$

20. Application of this expression (m) to determine how much the earth can be rotated by this force during the period of precession, measured in angle of circumference, remains now to be attained.

As already demonstrated from Fig. 4 for the earth taken of uniform density of mass (m)

$$d\,m = 3\,m\,y^2\,d\,y.$$

To make this result comply with the requirement of moment of force it must be multiplied by $r\,(1 - y^2)^{\frac{1}{2}}$ and this product integrated becomes

$$\int_0^1 3\,m\,r\,y^2\sqrt{1 - y^2}\,d\,y = \frac{3\,\pi}{16}\,r\,m.$$

For one rotation of the earth on polar axis this result multiplied by $2\,\pi$ becomes

$$\frac{3\,\pi^2\,r\,m}{8}.$$

For the earth increasing in density from *crust* to center per law already explained the expression becomes

$$\frac{3\ \pi^2}{8}\ (r\ m_1 + r_1\ m_2 + \text{etc.}).$$

Let the quotient of this result divided by r m be q, then the expression becomes

$$\frac{3\ \pi^2\ r\ m\ q}{8}.$$

Let c be the angle of rotation of the earth for period of precession (P) as determined by observation, then the expression becomes

$$(n) \qquad \frac{3\ \pi^2\ r\ m\ c\ q}{8}.$$

21. This expression (n) then equals (m) multiplied by P.

$$(o) \qquad P = \frac{225\ \pi^4\ M\ n\ r\ c\ q\ m}{32\ \sin\Phi\ \cos\Phi\ \ (M\ n + L^3)\ r^2\ e^2\ Q\ m}.$$

The Value of c for Permanent Axis of Rotation.

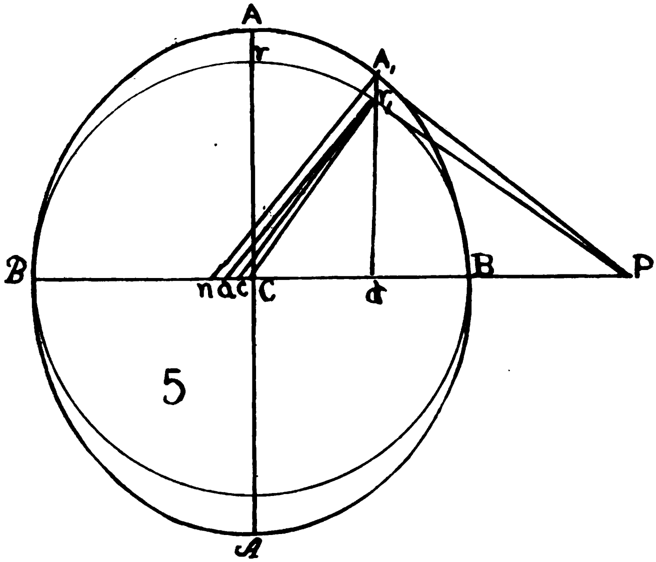
22. Take the earth to-day at point of vernal equinox, then 12,900 years hence the present point of summer solstice will be at present point of winter solstice, and the points of solstice and the pole of the equator and the pole of the ecliptic will be in same plane of to-day. During this half period of precession the pole of the equator will have shifted in same plane angle 2 Φ. It is likewise for return vibration. Truly taken thus

$$(p) \qquad c = \frac{4\ \Phi}{360} = \frac{\Phi}{90}.$$

The Value of $\frac{q}{Q}$ Determined.

23. In Fig. 5 B $\mathcal{B}$ and A $\mathcal{A}$ represent the polar and the equatorial

diameters of the earth. Line A_1 n perpendicular to tangent A_1 P is direction of normal. Radius of circle r_1 C is perpendicular to tangent r_1 P. Thus taken points A_1 and r_1 are located in ordinate A_1 d parallel to equatorial radius A C. Line r_1 a drawn parallel to normal A_1 n makes angle a r_1 C measure the direction of the normal from radius r_1 C. Let η represent this angle. Draw line r_1 c dividing angle η into two parts. Thus taken angle η is equal to the angle of the vertical. Let angle a r_1 c equal x and represent changed direction of attraction due to change of figure from sphere to spheroid. Under such conditions angle c r_1 C (c) is due to centrifugal force.



5

For radius unity and small eccentricity, sine, arc and tangent of angle η are materially equal.

Take angle r_1 C A=X.

The angle of the vertical is determined from surface eccentricity regardless of earth's interior density.

$$\eta = \sin \eta = \sin X \cos X \, e^2.$$

Let $\frac{1}{2} f e^2 =$ centrifugal force at the equator.

Then for latitude of point A_1 or r_1 centrifugal force acting in direction d r_1 equals $\frac{1}{2} f e^2 \cos X$. For change from direction d r_1 to direction C r_1, $\frac{1}{2} f e^2 \cos X$ for angle c becomes

$$c = \tfrac{1}{2} f e^2 \sin X \cos X.$$

$$x = \eta - c = (e^2 - \tfrac{1}{2} f e^2) \sin X \cos X = \tfrac{1}{2} e^2 (2 - f) \sin X \cos X.$$

This result for angle x is independent of the earth's interior density, and may be taken

$$r^2 m x = \tfrac{1}{2} r^2 m e^2 (2 - f) \sin X \cos X.$$

24. Take the earth made up, as already explained, of components, each component for itself of uniform density. Thus taken the value of x for each and every component can be obtained and the result for the whole earth determined. Thus determined the expression is

$$\tfrac{3}{5} (r^2 m_1 e + r_1^2 m_2 e_1^2 + \text{etc.}) \sin X \cos X = \tfrac{1}{2} r^2 m e^2 (2 - f) \sin X \cos X.$$

$$r^2 m_1 e^2 + r_1^2 m_2 e_1^2 + \text{etc.} = \tfrac{5}{6} (2 - f) e^2 r^2 m.$$

$$\text{(q)} \qquad \frac{r q m}{r^2 Q m e^2} = \frac{r m_1 + r_1 m_2 + \text{etc.}}{r^2 m_1 e^2 + r_1^2 m_2 e_1^2 + \text{etc.}} = \frac{\sqrt{\tfrac{5}{6}(2 - f)}}{\tfrac{5}{6}(r - f) e^2} = \frac{1}{\sqrt{\tfrac{5}{6}(2 - f)}\, e^2}.$$

The ratio of r to r^2 is $\sqrt{\tfrac{5}{6}(2 - f)}$ to $\tfrac{5}{6}(2 - f)$.

25. By due substitution of results (p) and (q) expression (o) becomes

$$\text{(r)} \qquad P = \frac{5 \pi^4 \Phi n}{64 \sin \Phi \cos \Phi \left(n + \frac{L^8}{M}\right)} \times \frac{\sqrt{\tfrac{5}{6}(2 - f)}}{\tfrac{5}{6}(2 - f) e^2}.$$

THE MASS OF THE MOON $\frac{1}{n}$ DETERMINED.

26. In the right-hand member of equation (r), all the factors are common for computing the precession due either to the sun or the moon excepting factor $\left(n + \frac{L^3}{M}\right)$. Term n is for the sun and term $\frac{L^3}{M}$ is for the moon. L^3 and M have constant values determined by observation. The ratio of precession due the sun to that due the moon can be determined from nutation caused by the moon.

Mean distance to sun $=$ 92,785,000 miles nearly.
Mass of sun $=$ 331,130.
Moon's mean distance $=$ 240,300 miles.

These constants of observation for sun and moon have the authority of Newcomb, and Newcomb and Holden.

The value of the semi-major axis of the ellipse of nutation is called the "constant of nutation," and it enters as an element into all reductions of astronomical observations.

"By 603 observations of Polaris, made at Dorpat between 1822 and 1838, M. Peters has determined the semi-axis major of the ellipse to be 9.2361″." (See Encyclopedia Britannica under head of Astronomy.)

"Very recently this important astronomical element has been investigated by Dr. Robinson, of Armagh, from the Greenwich observations with the mural circle from 1812 to 1835, and the result at which he arrives is 9.23913″." (See Memoirs Roy. Ast. Soc., Vol. XI, 1840.)

Duly computed

$$(s) \qquad n + \frac{L^3}{M} = n + 173.85.$$

27. For the line of the nodes of the moon's orbit taken in line of intersection of planes of equator and ecliptic, sin Φ cos Φ for points of solstice becomes for the moon

$$\sin(\Phi \pm i)\cos(\Phi \pm i).$$

$i =$ angle of inclination of moon's orbit to ecliptic.

As the moon's nodes retrograde, the pole of the moon's orbit measured from the pole of the ecliptic vibrates from angle zero to angle i. For resultant, then, measured in plane of the pole of the ecliptic and the points of solstice, expression $\sin (\Phi \pm i) \cos (\Phi \pm i)$ becomes

$$\sin (\Phi \pm i) \cos (\Phi \pm i) \cos i = \sin \Phi \cos \Phi \cos i$$
$$(1 - 2 \sin^2 i) \pm \sin i \cos^2 i \, (1 - 2 \sin^2 \Phi).$$

For precession for a full nodical period $\pm \sin i \cos^2 i \, (1 - 2 \sin^2 \Phi)$ vanishes and becomes an expression for nutation.

Integrated from zero to $5° \; 8' \; 48''$

(t) $$\int \sin \Phi \cos \Phi \cos i \, (1 - 2 \sin^2 i) \, di = .993337 \sin \Phi \cos \Phi.$$

This expression is for outstanding correction to determine the true precession for the moon, and not to be used for expression to obtain moon's mass.

28. Expression $\pm \sin i \cos^2 i \, (1 - 2 \sin^2 \Phi) \, di$ integrated for due limits becomes

$$.030513.$$

Since $\sin \Phi \cos \Phi = .36568.$

(u) $$.030513 = .083445 \sin \Phi \cos \Phi.$$

Sign $\pm$ means simply advance and return vibrations for nodical period of nutation. Result (u) is for minor axis of ellipse of nutation.

29. For the major axis of this ellipse take points of equinox from which to measure the vibrations of the moon's orbit in plane of pole of ecliptic and points of equinox. This vibration for one-half the nodical period is plus angle i, varying from zero to angle i, and for the other half minus. As plus and minus in this case simply indicate opposite direction for full period of nutation we can use $\sin i \cos i$ under

the conditions of $\sin \Phi \cos \Phi$ developed from $\sin \theta \cos \theta$ in case of precession.

The recession of the moon's nodes is measured from points of equinox. The angle of inclination of planes of the equator and moon's orbit varies from angle $(\Phi + i)$ to $(\Phi - i)$. The cosine of this varying angle $(\Phi \pm i)$ as measured from the pole of the ecliptic in the plane of the pole of the ecliptic and points of solstice is the true modifier for $\sin i \cos i$. Thus taken for the two conditions considered, the differential expression becomes

$$\sin i \cos i \cos (\Phi \pm i)\, di = \cos \Phi\, (\sin i - \sin^3 i)\, di.$$

For limitation zero and angle 5° 8′ 48″

$$\text{(v)} \qquad \int \cos \Phi\, (\sin i - \sin^3 i)\, di = .040995 = .112106 \sin \Phi \cos \Phi.$$

It is now evident that .083445 and .112106 determine the ratio for minor and major axes of the ellipse of nutation.

30. For the period of precession taken 25,800 years, the component force of precession caused by the sun and moon is represented as already determined by 94 degrees. For the nodical period 18.5997 years the computed result is 243.96″.

$$243.96'' \times .112106 = 27.3494''.$$

Under the hypothesis that the sun as well as the moon causes nutation, 27.3494″ should be the major axis, instead of 18.4782″.

27.3494″ — 18.4782″ = 8.8712″ represents under the hypothecated term of nutation for the sun.

$$\frac{18.4782}{8.8712} = 2.083.$$

31. As thus determined the precession caused by the moon is 2.083 times greater than that for the sun.

In equation (s) term n is for the sun and 173.85 for the moon. Thus taken

$$2.083\ n = 173.85.$$
$$n = 83.46.$$
$$\frac{1}{n} = \frac{1}{83.46}.$$

To obtain true value for n, the moon and sun must be taken under conditions of same value for angle Φ and angle $(\Phi \pm i)$, and for same eccentricity of orbits. As computed by equation (t) having factor $\sin \Phi \cos \Phi$, n must be multiplied by .993337 on account of smaller mean value for $\sin (\Phi \pm i) \cos (\Phi \pm i)$, and this product multiplied by .9836 for greater eccentricity of moon's orbit. (See Section 58 of this publication.)

Duly computed for these two corrections

$$n = 81.54.$$

$$\text{(w)} \qquad \frac{1}{n} = \frac{1}{81.54}.$$

32. This result for the moon's mass is correct, providing the constants of observation used for computation are true. The same result for the moon's mass is attained from the minor axis of the ellipse of nutation.

By the mural circle of the observatory very exact results can be obtained for the moon's declination or latitude, or for the major axis of the ellipse of nutation.

THE EARTH'S ELLIPTICITY.

33. Having determined the moon's mass, the mean surface ellipticity of the earth can be obtained from equation (r). Before using this equation for that purpose it must be corrected.

To determine the true period of precession, the lunar term of equation (r) must be diminished for mean value for angle $(\Phi \pm i)$, being less than that for Φ, and for greater eccentricity of moon's orbit.

Thus taken, and due computation made, equation (r) becomes

$$P = \frac{5 \times 81.54\ \pi^4\ \Phi}{64 \times 251.44 \sin \Phi \cos \Phi} \times \frac{1}{\sqrt{5/6\,(2 - f)\,e^2}}.$$

Taking $\Phi = 23\tfrac{1}{2}$ degrees

$$\text{(x)} \qquad P = 158.61 \times \frac{1}{\sqrt{5/6\,(2 - f)\,e^2}}.$$

$P = 25800$ years as determined by observation.

As already taken

$\tfrac{1}{2} f\, e^2 =$ Centrifugal force at the earth's equator.

The value of this force is not far from fraction .0034580.

Thus taken equation (x) for value in term e^2 becomes

$$e^4 - .003458\ e^2 = .000022675.$$

$$e^2 = .006795.$$

34. For second order of eccentricity (e^2) and first order of ellipticity (ϵ)

$$e^2 = 2\,\epsilon.$$

Thus taken

$$\text{(y)} \qquad \epsilon = .0033975 = \frac{1}{294.34}.$$

35. The result attained for the mean ellipticity of the earth by Elias Loomis, LL.D., late Professor of Natural Philosophy and Astronomy of Yale College, is $\frac{1}{294}$. He obtained this result from a mathematical discussion of arc measurements made on both continents. (See Loomis' Treatise on Astronomy.)

$\frac{1}{294.98}$ is used by the U. S. Coast Survey Department and it has been found very closely true for the whole territory of the United States. Thus tested $\frac{1}{294.98}$ is a very reliable result of observation. (See Report of U. S. Coast Survey, etc., 1884, Appendix No. 6.)

These two results of observation are a very good test of our result $\frac{1}{294.34}$ obtained deductively from the Newtonian law of attraction.

36. The close agreement of our computed and the observed results for ellipticity much more than suggest that the material composing the earth, as now located and adjusted, is essentially under condition of fluid equilibrium, and that the mountains and the continents float above the level of the sea under the law for the floating of icebergs. It is thus evident that the part of the *crust* of the earth embracing the continents and that part covered by oceans are so poised that a material washdown from the land into the sea causes the sea bed to sink and the land surface to rise again. Owing to the inflexibleness of the earth's crust, a readjustment to a state of equilibrium is sometimes attended by an earthquake. Such statement for balance is the theory of the geologist, and this theory for geology is at least strongly reinforced by our mathematical determination of the earth's mean ellipticity.

37. In terms of ellipticity in place of eccentricity, for centrifugal force at the earth's equator

$$\tfrac{1}{2} f\, e^2 = f\, \epsilon = .0034580.$$

$$f = 1.017807.$$

$$\sqrt{\tfrac{5}{6}\ (2-f)} = .904706.$$

$$\tfrac{5}{6}\,(2 - f)\,\epsilon = .818494\,\epsilon = .0027808.$$

$$(z) \qquad \frac{r\, m_1 + r_1\, m_2 + \text{etc.}}{r^2\, m_1\, \epsilon + r_1^2\, m_2\, \epsilon_1 + \text{etc.}} = \frac{.904706}{.818494\epsilon}.$$

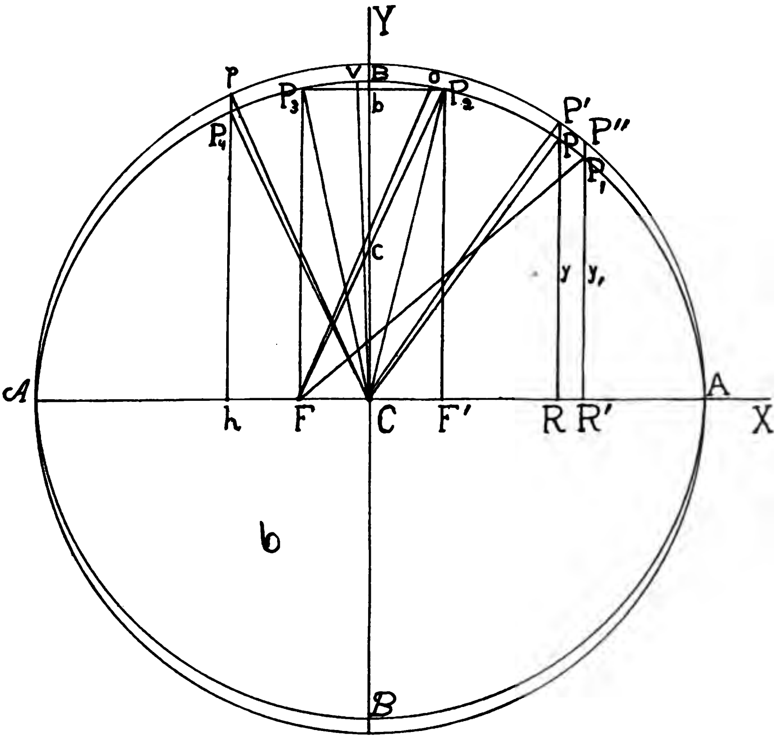
38. The relation of the terms of the first member of this equation

is such that the true value for the second member of the equation can be reached by using assumed laws innumerable, for increase of the earth's density from surface to center. These assumed laws determine nothing more than our expression (q).

There is a law attainable from well-known data which truly determines the density of the earth from surface to center. By this law determined, the mean crust density for thirty miles of thickness is not far from three specific gravity, and the central density of the earth a little more than ten. The specific gravity of the surface layer of the crust of the earth, as determined by observation, is not far from that of granite rock. As the author mathematically investigates the problem taken under all its essential conditions for correct solution, it does not require the interior component particles of the earth below the crust, for stable equilibrium against the tidal force, to be bound together by the tensile strength of steel bars. In other words, any under crust particle may have some freedom of independent motion. Further remark here on this question is outside the prescribed limits of this publication.

OUTSTANDING CORRECTIONS DUE ELLIPTICAL ORBIT.

39. In Fig. 6, let A $\mathcal{A}$ be major axis of the ellipse, having principal axes A $\mathcal{A}$ and B $\mathcal{B}$, and F and F^1, the foci. Also let A $\mathcal{A}$ be the diameter of the circumscribed circle. And also let C P^1 be any radius ofthe circle having angle P^1 C A represented by z.



The area of the circumscribed circle can be divided into equal parts by a system of radii having equal angles at the center from radius to radius or dz angles. The area of the ellipse can also be divided into

equal parts by a system of semi-diameters, each semi-diameter of the system having the same relation to a corresponding radius of the system of the circle that semi-diameter C P has to radius C P^1. Otherwise stated, for division of the ellipse into equal parts ordinate P R of the ellipse must be a part of ordinate P^1 R of the circle.

40. Kepler's first law, that the radius vector of a planet describes about the sun equal areas in equal times, requires for the purpose of this demonstration to divide the area of the ellipse into equal parts by a system of radius vectors for focus F.

41. For the first quadrant.

Ordinate P^1 R of the circle equals

$$a \sin z.$$

Ordinate P R (y) of the ellipse equals

$$b \sin z = a\,(1 - e^2)^{1/2} \sin z.$$

In these expressions a and b equal the semi-major and semi-minor axes, and e the eccentricity of the ellipse.

42. Now take point P_1 of first quadrant so that its ordinate

$$y_1 = a\,(1 - e^2) \sin z.$$

Likewise for second quadrant

$$y_{11} = a\,(1 - e^2) \sin z.$$

For angle z taken 90 degrees y_1 becomes semi-parameter P_2 F^1 and y_{11}, semi-parameter P_3 F.

It is evident that the area of the first quadrant of the ellipse less one-half of segment P_3 B P_2 equals the area of the second quadrant less the other half of same segment. Such equality is true for any value of sin z for z less than 90 degrees.

43. As triangle c b P_2 equals triangle c C F, the area of that part of the semi-ellipse to the right hand of radius vector P_2 F less one-half of segment P_3 B P_2 equals that to the left less the other half of same segment. Now take point o at some point between P_2 and B so that the radius vector o F shall divide the area of the semi-ellipse into equal parts. Under such conditions the area of sector o F P_2 equals one-half of segment P_3 F P_2.

44. Take point v so that sector v C B shall equal sector o F P_2. Under this condition for radius unity we must integrate angle z for limit $\frac{\pi}{2}$ + arc v B, to make radius vector o F divide the semi-ellipse into two equal parts. And for the second quadrant we must integrate for angle z between limits zero and $\frac{\pi}{2}$ — arc v B.

The area of sector v C B can be taken as a standard unit measure for measuring the area for the whole ellipse. The area of this sector can also be used for measuring periodic time. It is evident when we integrate for angle z for the first quadrant for limits zero and $\frac{\pi}{2}$ + arc v B we integrate for more than one-fourth of the area of the ellipse as divided into four equal parts by principal axes A $\mathcal{A}$ and B $\mathcal{B}$ by area of sector v C B. To correct for this we must lessen the result as determined by such integration by area of sector v C B. It is likewise for the third and fourth quadrants.

45. Let x = C R^1.

Take y for y_1 or y_{11}. Then as so modified

$$y = a\ (1 - e^2)\ \sin z.$$

Thus taken, per the equation of the ellipse

$$x = a \sqrt{1 - (1 - e^2) \sin^2 z.}$$

$$C\ F = C\ F^1 = a\ e.$$

For the first quadrant let radius vector F P_1 = r.

For the second quadrant take r_1 for the radius vector.

$$r^2 = (x + a\ e)^2 + y^2.$$

$$r_1^2 = (x - a\ e)^2 + y^2.$$

True for e^2, differential expression for

(A) $$\frac{1}{r^2} = \frac{1}{a^2}\,(1 - 2\ e \cos z\ dz + 3\ e^2 \cos^2 z\ dz).$$

(B) $$\frac{1}{r_1^2} = \frac{1}{a^2}(1 + 2\ e \cos z\ dz + 3\ e^2 \cos^2 z\ dz).$$

(C) $$\frac{1}{r^3} = \frac{1}{a^3}\,(1 - 3\ e \cos z\ dz + 6\ e^2 \cos^2 z\ dz).$$

(D) $$\frac{1}{r_1^3} = \frac{1}{a^3}\,(1 + 3\ e \cos z\ dz + 6\ e^2 \cos^2 z\ dz).$$

In these equations $\frac{1}{r^2}$, $\frac{1}{r_1^2}$, etc., represent values so as to make the two members of equation true.

46. As determined by observation the eccentricity of the earth's orbit is .016771, and that of the moon's orbit .05491, and that of planet Mercury's orbit .20560.

Per scale for draft of Fig. 6, focal distance C F is for planet Mercury, or is drawn more than 12 ¼ times too great for the earth's orbit. For the earth's orbit arc P_3 B P_2 of the ellipse can be taken as arc of circle of radius C B without material error, and focal distance C F can be used for arc B P_2, or arc B P_2 taken equal to ae.

$$\tfrac{1}{2} \text{ chord } P_3\,P_2 = b\ P_2 = a\ e.$$

$$\text{Arc B } P_2 = \text{arc for a e.}$$

$$\text{The area of } \tfrac{1}{2} \text{ sector } P_3\,C\,P_2 = \frac{a\ b \text{ arc for } e}{2} = \frac{a\ b\ e}{2}.$$

$$\text{Area of triangle C b } P_2 = \frac{a\ b\ e}{2}\sqrt{1 - e^2}.$$

Area of ½ segment $P_3 B P_2 = \frac{a b e}{2} (1 - \sqrt{1 - e^2}.)$

Since $\sqrt{1 - e^2}$ equal cos e, area of ½ segment is

$$\frac{a b}{2} (1 - \cos e) e = \text{area of sector v C B}.$$

$$\frac{a b \pi}{2} \times \frac{(1 - \cos e) e}{2} = a b \pi \frac{(1 - \cos e) e}{4}.$$

$a b \pi$ is the area of the ellipse. Measured by the area of the whole of the ellipse the area of sector v C B equals $a b \pi \frac{(1 - \cos e) e}{4}$.

Due to the very small eccentricity of the earth's orbit the area of sector v C B equals ½ of arc v B × b, or for b taken unity, the area of the sector equals ½ of arc v B. For the whole ellipse arc v B then may be taken $\frac{(1 - \cos e)}{2}$ e.

The angular velocity of the earth in orbit varies inversely as square of radius vector. Take arc z_1 equal arc $z + \frac{(1 - \cos e) e}{2}$ for the first quadrant, and arc z_2 equal arc $z - \frac{(1 - \cos e) e}{2}$ for the second quadrant.

Our expression (A) for $\frac{1}{r^2}$ for first quadrant becomes

$$\text{(E)} \qquad \frac{1}{r^2} = \frac{1}{a^2} (1 - 2 e \cos z_1 \, dz_1 + 3 e^2 \cos^2 z_1 \, dz_1).$$

And for the second quadrant

$$\text{(F)} \qquad \frac{1}{r_1^2} = \frac{1}{a^2} (1 + 2 e \cos z_2 \, dz_2 + 3 e^2 \cos^2 z_2 \, dz_2).$$

47. For the earth's orbit

$$e = .016771.$$

$$\frac{(1 - \cos e) e}{2} = .0000011794 = .2434''.$$

The integrated result for expression cos z *dz* is

$$\cos \overline{z} = \frac{\sin z}{\operatorname{arc} z}.$$

For z varying from zero to $\frac{\pi}{2}$

$$\cos \overline{z} = \frac{2}{\pi} = .6366197724.$$

48. For angle

$$z_1 = \frac{\pi}{2} + .0000011794 = 90^\circ\ 0'\ .2434''.$$

$$z_2 = \frac{\pi}{2} - .0000011794 = 89^\circ\ 59'\ .3566''.$$

It is evident that for obtaining an outstanding correction amounting to only about .0074″ for one revolution of the earth around the sun for period of one year that the sine for arc $\frac{\pi}{2} + .0000011794$ can be taken unity without material error.

Taken thus between limits zero and $\frac{\pi}{2} + .0000011794$

$$\cos \overline{z_1} = \frac{1}{\frac{\pi}{2} + .0000011794} = .6366192943.$$

And for limits zero and $\frac{\pi}{2} - .0000011794$

$$\cos \overline{z_2} = \frac{1}{\frac{\pi}{2} - .0000011794} = .6366202520.$$

49. Area of sector v C B $= \frac{.0000011794}{2} = .0000005897.$

This result is for the area of the ellipse of the earth's orbit taken unity. Taken thus, fraction .0000005897 can be used as a standard unit measure as before stated for measuring periodic time for one year

in case of the earth. Due to this fractional excess of periodic time, our result .6366192943 for the integration of expression cos $z_1\,dz_1$ is too great by excess .0000005897. The true result for this integration then is

(G) $\cos \overline{z_1} = .6366187046.$

Likewise for equal deficiency of periodic time the true integration of expression cos $z_2\,dz_2$ is

(H) $\cos z_2 = .6366208417.$

For the orbit of a circle or for the sun at the center of the earth's elliptic orbit instead of at focus F, the expression is

$$\frac{2}{\pi} = .6366197724.$$

The due outstanding correction to be made to the aphelion half of radius vectors then is

$$.6366187046 - .6366197724 = -.0000010678.$$

Likewise computed for the perihelion half of radius vectors, the result is

$$.6366208417 - .6366197724 = .0000010693.$$

50. For outstanding correction terms 1 of expressions (E) and (F) are not to be considered, and for terms $3\ e^2 \cos^2 z_1\,dz_1$ and $3\ e^2 \cos^2 z_2\,dz_2$ the correction is too small to be material. The due outstanding correction for terms $-\ 2\ e \cos z_1\,dz_1$ and $2\ e \cos z_2\,dz_2$ becomes

$$-\ 2\ e \times -\ .0000010687 = 2\ e \times .0000010687,$$

and

$$2\ e \times .0000010693.$$

Thus taken the mean of the sum of the two is

(I) $2\ e \times .0000010685.$

This result determines the advanced position of the earth in orbit beyond that otherwise computed for one revolution, or one year.

For eccentricity .016771 for the earth's orbit

$$2\ e \times .0000010685 = .00000003584 = .0074''.$$

And for a century .74″.

51. Per draft of Fig. 6, angle B C P_2 is about 12°, or that due planet Mercury. For the moon's orbit this angle is about 3° 8′, and for the earth's orbit about 58′. For angle 58′ arc P_3 B P_2 can be taken for arc of circle having B C for radius without material error. But for the moon and planet Mercury it cannot be so taken. Besides this for 58′, sine, chord and arc are of materially the same length. For planet Mercury and others of the planets and for the moon another step of demonstration can be made and another expedient used.

52. For expedient take planet Mercury's mean distance from the sun the same as that for the earth and both revolving around the sun in the same plane. Under such condition the periodic times and the areas of orbit for the two would be the same excepting for the very small outstanding corrections. Per Fig. 6, the two orbits from B toP_2 would very nearly coincide. Taken then for 58′ of arc to the right hand of B or for the true position of semi-parameter P_2 F^1 for the earth's orbit regardless of the true position for the parameter for the orbit of Mercury, it is evident that sector v C B for arc 58′ for both orbits would have materially the same orbital area. Taken thus to correct the resulting error would require an additional outstanding correction to a very small outstanding correction.

The integrated values of the factors to be substituted for cos $z_1\,dz_1$ and cos $z_2\,dz_2$ in terms 2 e cos $z_1\,dz_1$ and 2 e cos $z_2\,dz_2$ of expressions (E) and (F) are the same for the Earth and Mercury taken under the conditions stated. The outstanding corrections then for the two planets for one revolution each, are in ratio of their eccentricities.

The eccentricity of the orbit of Mercury is 12.261 times greater than that for the earth and must be so taken for expressions (E) and

(F). For mean distance of the earth's orbit, then, Mercury's gain in motion for the cause under consideration for one such revolution would be

$$.0074'' \times 12.261 = .09073''.$$

Mercury in its own orbit makes 4.152 revolutions per year. For a century then the result is

$$.09073'' \times 4.152 \times 100 = 37.67''.$$

53. Simon Newcomb in his "Popular Astronomy" says: "From a discussion of all the observed transits of this planet (Mercury) across the disk of the sun, Leverrier has found that the motion of the perihelion of Mercury is about 40 seconds (of arc) in a century greater than that computed from the gravitation of other planets." Our computed result 37.67″ and the observed result of about 40″ are in close agreement when duly considered for errors of observation and a slightly faulty system of reduction due to tremulous motions of the earth.

54. As determined by the method of computation, as already explained for the planet Mercury, the other planets are ahead of their otherwise computed positions, for a century as follows: Venus .4908″, the Earth .74″, Mars 2.19″, Jupiter .179″, Saturn .0839″, Uranus .0235″ and Neptune .00241″.

THE OUTSTANDING CORRECTION FOR THE MOON.

55. Measured by attraction, the attraction of the sun on the earth is 2.20 times greater than that of the earth on the moon. And as measured in terms of periodic time instead of distance this 2.20 becomes the square root of the cube of 2.20, or 3.263. The eccentricity of the earth's orbit taken unity, the moon's eccentricity becomes 3.274. The moon makes 13.37 revolutions per year.

The product of $.0074'' \times 3.263 \times 3.274 \times 13.37 = 1.06''$ nearly. The moon then for one year is ahead of its otherwise computed place 1.06″.

This result very closely meets the requirements of observation as explained in our introduction. And this outstanding correction makes valid the Newtonian law of attraction for one of the cases heretofore considered exceptional.

REMARKS.

56. In this discussion, by taking advantage of expedients, a complexity of demonstration is avoided, and more accurate results attained than could otherwise be obtained. The author has verified the results of this investigation by several independent systems of demonstration. For e^3 and higher orders of eccentricity the integration of tan $z\,dz$ for limitation zero and $\frac{\pi}{2}$ is required. Besides this there are about 4800° for 13.37 revolutions of the moon, or 17280000″. By the method of expedients for obtaining outstanding correction only about 1″ is involved instead 17280000 + 1. For the 1″taken alone after eliminating the seventeen million two hundred and eighty thousand seconds, material accuracy for outstanding correction is reached by integrating the differential factors for eccentricity first power.

OUTSTANDING CORRECTION DUE THE FORCES OF PRECESSION.

57. Precession caused by the sun and the moon varies directly as mass and inversely as cube of distance. To determine then these outstanding corrections equations

$$\text{(C)} \qquad \frac{1}{r^3} = \frac{1}{a^3}\,(1 - 3\,e\cos z\,dz + 6\,e^2\cos^2 z\,dz)$$

and

$$\text{(D)} \qquad \frac{1}{r_1^3} = \frac{1}{a^3}\,(1 + 3\,e\cos z\,dz + 6\,e^2\cos^2 z\,dz)$$

can be used.

First.—To determine outstanding correction for the moon's mass.

Second.—Outstanding corrections for the vibrations of the earth's polar axis.

Third.—To determine the slow but continual decrease of the angle of obliquity of the ecliptic.

First.—

58. In equation (D) terms $3\ e \cos z\ dz + 6\ e^2 \cos^2 z\ dz$ are terms for an advance or plus outstanding correction for vibration of the earth's polar axis, and likewise in equation (C) terms $-3\ e \cos z\ dz + 6\ e^2 \cos^2 z\ dz$ are for a due return or minus vibration. Result $\frac{\Phi}{90}$ of expression (p), page 28, is for the sum of an advance and a due return vibration of the earth's polar axis for a full period of precession. The outstanding correction for $\frac{1}{r_1^3} + \frac{1}{r^3}$ must then be taken $12\ e^2 \cos^2 z\ dz$.

$$\int_0^1 12\ e^2 \cos^2 z\ dz = 6\ e^2.$$

This result, $6\ e^2$, is materially accurate for the correction of the moon's mass for the moon's orbit of an ellipse.

Second.—

59. Duly integrated expression $\left(\frac{1}{r^3_1} - \frac{1}{r^3}\right)$ for the vibration of the earth's polar axis becomes

$$\frac{12}{\pi} e.$$

This result is not far from accurate for heading (Second) for expression in terms of diameter of circle. But it determines nothing for heading (Third). For this reason it is best to readjust expression for both headings.

60. Per Fig. 6, the area of sector v C B is too small to have material consequence in the case for e^2. Besides this, expressions for $\frac{1}{r^3}$ and $\frac{1}{r_1^3}$ each have terms $6\ e^2 \cos^2 z\ dz$ plus.

Expression for $\frac{1}{r^3}$ is for aphelion or apogee radius vectors. For precession all radius vectors to the right hand of semi-parameter $P_3 F$ are for aphelion or apogee distances.

61. For Fig. 6, conceive ordinate $F\ P_3$ extended to circumscribed

circle. Let arc of this circle measured from co-ordinate Y to point of its intersection with ordinate F P_8 extended, be represented for radius unity by arc e. Take point p, 2 arc e distance measured from co-ordinate Y, and draw radius C p. Then angle $\mathcal{A}$ C p is a (z angle). Under conditions already explained for the first quadrant let

$$y_8 = a\ (1 + e^2)\ \sin z = P_8\ F \text{ or } \tfrac{1}{2} \text{ of parameter.}$$

For radius unity it is now evident that all ordinates for y_8 within the due limits of e can be taken cos e. Thus taken arc for angle X C p equals $\frac{\pi}{2} + 2$ e, and arc for angle $\mathcal{A}$ C p equals $\frac{\pi}{2} - 2$ e.

62. Per Fig. 6, there is greater area and more radius vectors to the right hand of semi-parameter P_8 F than to the left by area 2 a e $\times$ b, or for radius unity area 2 e.

As already demonstrated for the first quadrant parameter P_2 F^1 is too far to the right hand by arc o P_2 to divide the area of the semi-ellipse into two equal parts. But radius vector o F makes an equal division. For the same reason parameter P_8 F is too far to the right by arc equal to oP_2.

From reasoning of previous demonstration, y_1 taken equal to a $(1 - e^2)$ sin z is a trifle too small. By same reasoning, y_8 taken equal to a $(1 + e^2)$ sin z is too large by same trifle. Adjustment to true values for y_1 and y_8 leads to complexity of solution. To avoid such complexity and perplexity an expedient can be used.

63. The difference between the number of radius vectors for aphelion distances to the right hand of semi-parameter P_8 F and perihelion radius vectors to the left is represented, for radius unity, by area e or integration of differential of arc 2 e de. When we integrate expression cos z dz for limitation zero and $\frac{\pi}{2}$ we integrate for all the radius vectors and the area to the right hand of radius vector P_2 F. Also when we integrate expression arc 2 e de we integrate for the radius vectors between radius vector P_2 F and radius vector P_8 F.

Under such conditions true for the first order of eccentricity, the integration of expression (C) becomes

$$\text{(E)} \qquad \frac{1}{r^3} = \frac{1}{a^3}\left(1 - \frac{6\,e}{\pi} - 3\,e \operatorname{arc} e\right).$$

When we integrate expression $\cos z\, dz$ for limitation zero and $\frac{\pi}{2} - 2\,e$, we integrate for all the radius vectors and the area to the left hand of radius vector $P_3\,F$. Taken thus

$$\cos \overline{z} = \frac{\sin\left(\frac{\pi}{2} - 2\,e\right)}{\frac{\pi}{2} - 2\,e}.$$

The due integration of expression (D) then becomes

$$\text{(F)} \qquad \frac{1}{r_1^3} = \frac{1}{a^3}\left(1 + 3\,e \times \frac{\sin\left(\frac{\pi}{2} - 2\,e\right)}{\frac{\pi}{2} - 2\,e}\right).$$

64. For the earth's orbit

$$e = .016771.$$

For this eccentricity of orbit equations (E) and (F) by due computation become

$$\frac{1}{r^3} = \frac{1}{a^3}(1 - .032874).$$

$$\frac{1}{r_1^3} = \frac{1}{a^3}(1 + .032709).$$

$$\frac{1}{r_1^3} - \frac{1}{r^3} = \frac{1}{a^3} \times .0656 \text{ nearly.}$$

For such very small outstanding correction we can take perihelion distance of sun at time of winter solstice with material accuracy, or otherwise stated, take radius vector of January first for radius vector of preceding December twenty-first.

The total angular change of the earth's polar axis, as caused by the attraction of the sun and the moon, from December twenty-first to following December twenty-first or for time one year, for orbits of

circles is 13.1″. That part of this 13.1″ due the sun is 4.25″; and that part due the moon is 8.85″. The change in latitude for 1″ is 101 feet.

$$101 \times 4.25 \times .0656 = 28 \text{ feet, nearly, for the year.}$$

Measured by diameter of circle the north and the south pole of the earth each revolve in a so-called circle 28 feet in diameter during the year and return to starting point at the end of the year. This circle 28 feet in diameter is due to the varying distance of the sun, or to the taking of the earth's orbit an ellipse instead of a circle as usually taken for precession and nutation, and to the spheroidal figure of the earth.

65. Assume the moon revolving around the earth in the plane of the ecliptic, and take the perigee of the moon's orbit at point of winter solstice.

For the moon's orbit

$$e = .054908.$$

$$\frac{1}{r^3} = \frac{1}{a^3}(1 - .1139112.)$$

$$\frac{1}{r_1^3} = \frac{1}{a^3}(1 + .1120710).$$

$$\frac{1}{r_1^3} - \frac{1}{r^3} = \frac{1}{a^3} \times .226.$$

This result is for an anomalistic revolution. There are nearly 13.3 such revolutions of the moon in one year. As already stated there is due the moon for precession computed for the orbit of a circle for one year 8.85″. And for one month the result is .666″.

$$101 \times .226 \times .666 = 15.2 \text{ feet.}$$

66. In one year the moon's perigee advances from point of winter solstice towards the point of vernal equinox or from west to east about 40.68°. For point of this 40.68° angle the diameter of so-called circle for a single revolution of the moon is less than 15 feet; and its true value can readily be computed. For perigee at point of vernal or autumnal equinox the diameter of circle is zero, and for per-

igee at point of summer solstice the result is the same as that for winter solstice. The causes for these diameters of so-called circle are the same as already stated for the sun's attraction on the component particles of the earth.

67. Again for the moon's attraction, other discussion is required for the moon's inclination of orbit to the plane of the ecliptic. The so-called ellipse of nutation is caused by this angle of inclination. As determined by observation the axes of this ellipse are 18.47″ and 13.74″. These numbers are mean results determined from many observations made during many years. In other words these results thus obtained are practically for a circular orbit for the moon. Nutation is materially due to the attraction of the moon.

68. The nodical period of the moon is about 18.6 years and the nodes move from east to west receding about 19.35 degrees per year. The perigee of the moon's orbit makes a complete revolution in about 8.85 years, advancing from west to east about 40.68 degrees per year. Take perigee at point of winter solstice, and also take the descending node at place of greatest descent, at same point. At the end of one year the descending node and place of perigee are 60 degrees apart, or, expressed in other words, $\frac{1}{6}$ of a year. The 19.35 degrees for the node measured from east to west or 340.65 degrees measured from west to east means one year. By this nodical standard of measure the advance of the perigee 40.68 degrees from west to east from point of winter solstice must be taken $\frac{7}{6}$ years or 426 days. As the 365 $\frac{1}{4}$ days and the 426 days determine the ratio for the movements of the node and the perigee, and as the question at issue depends on the positions of the node and the perigee, 426 days can be taken for a true period. How Chandler from a discussion of the data of observation ever reached his period of 428 days so close to the result as determined by the Newtonian law of attraction is a conundrum he can best answer.

69. For the period 426 days as computed from the major axis of

the ellipse of nutation the result is 26 ⅔ feet. And likewise computed from the minor axis the result is 20.1 feet. Thus determined the north and south pole of the earth, each revolve in an ellipse having major axis 27 feet nearly and minor axis 20. This ellipse is similar to the ellipse of nutation and its major axis extends in direction to the pole of the ecliptic.

For the two superposed revolutions of the poles of the earth as caused by the attraction of the moon taken in combination, the maximum result is 42 feet and the minimum zero.

70. For the three superposed revolutions of the poles of the earth taken in combination, there results as C. A. Young says, "a veritable 'wabble,' as is produced by striking a spinning top." By due computations made under the conditions of the requisite astronomical data, the wabbles for the earth's poles for every day for years in advance can be determined, and the maximum result from such compution cannot exceed 70 feet and the minimum not less than zero.

71. There is another computed period of 7 years derived from periods 365 ¼ and 426 days. This seven years' period is of no material worth for making computations for outstanding corrections for the nautical almanac.

$$365\ {}^1\!/_7 \text{ days} \times 7 = 426 \text{ days} \times 6.$$

This numerical equation does not include the anomalistic monthly period having maximum diameter of circles varying from 15 feet to zero.

There is also another period

$$18.6 \times 8.8 = 164 \text{ years nearly.}$$

72. For the demonstration of expression sin θ cos θ sin b db, (section 16, page 25,) line representing sin b is perpendicular to line of intersection of planes of equator and ecliptic. For corresponding line taken parallel to this line of intersection sin b becomes cos b, and

the expression can be taken sin θ cos θ cos b db. In terms of angle Φ and angle z

$$\sin\theta\cos\theta\cos b\,db = \sin\Phi\sin z\cos z\,dz.$$

These expressions explain and determine the value of the *so-called circle.*

Third.—

73. As already demonstrated and computed for the moon's orbit

$$\frac{1}{r^3} = \frac{1}{a^3}(1 - .1139112).$$

$$\frac{1}{r_1^3} = \frac{1}{a^3}(1 + .1120710).$$

In these two equations plus and minus before terms .1120710 and .1139112 can be taken for directions of vibration. The difference, then, in length of a forward and a return vibration is .0018402.

The vibration of the earth's polar axis due the moon for precession for one year, as before stated, is 8.85″.

$$8.85'' \times .00184 = .016284''.$$

Likewise computation for the sun's attraction on the earth for one year is

$$4.25'' \times .000165 = .000701''.$$

The total for the sun and moon = .016985.

Thus taken it would require nearly 59 years to decrease the angle of obliquity of ecliptic 1″. Likewise computed for angle of obliquity 45 degrees it requires 43 years for 1″ decrease. For the mean of the two, the result is about 50 years for 1″ decrease. Under the assumption that the eccentricities of the orbits of the earth and the moon have been the same in past time as at present, and will remain so for the